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Functions of Surfaces with Exceptional Points or Curves.*

BY CHARLES A. FISCHER.

In a former paper† I have given a definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a line, and have proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the double integral of the derivative multiplied by the first variation of the dependent variable. The object of the present paper is to extend the theory to the case where there are points or curves where the derivative does not exist. In the first section exceptional points are considered, and an application made to the second variation of a function of a double integral. In the second section exceptional curves are discussed, and in the last an application is made to the variable boundary problem of the calculus of variations.

§ 1. *Exceptional Points.*

It will be assumed that the derivative $L'(S; x, y)$ of a function $L(S)$ is continuous and approached uniformly with order $r \dagger$ at every point of the surface

$$S: \quad z = z(x, y),$$

excepting at the point (x_0, y_0) . The surfaces

$$S_\epsilon: \quad z = z(x, y) + \eta(x, y),$$

and

$$S_\alpha: \quad z = z(x, y) + \omega(x, y, \alpha),$$

will be defined as in the former paper.‡

Two kinds of exceptional points will be considered. The first kind will be those for which the inequality

$$\lim_{\epsilon=0} \frac{L(S_\epsilon) - L(S)}{\alpha} = 0, \tag{1}$$

is satisfied for every choice of $\eta(x, y)$ and α such that $\eta(x, y)/\alpha$ is bounded everywhere, and at (x_0, y_0)

* Read before the American Mathematical Society, August 3, 1915.

† Fischer, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVI (1914), p. 289.

‡ It can easily be proved that if $L(S)$ is continuous, and $L'(S; x, y)$ is approached uniformly, it is also continuous in all arguments. Compare with Evans, *Bulletin of American Mathematical Society*, Vol. XXI (1915), p. 389.

§ Fischer, *loc. cit.*, pp. 290-291.

$$\lim_{\epsilon=0} \frac{1}{\alpha} \cdot \frac{\partial^{i+j} \eta(x_0, y_0)}{\partial x^i \partial y^j}, \quad (i+j \leq r),$$

exists. The second kind will be those for which there is a constant a_0 such that

$$\lim_{\epsilon=0} \frac{L(S_\epsilon) - L(S)}{\alpha} = a_0 \lim_{\epsilon=0} \frac{\eta(x_0, y_0)}{\alpha},$$

or in a more general case those for which there is a set of constants a_{ij} such that

$$\lim_{\epsilon=0} \frac{L(S_\epsilon) - L(S)}{\alpha} = \sum_{i,j=0}^n a_{ij} \lim_{\epsilon=0} \frac{\partial^{i+j} \eta(x_0, y_0)}{\partial x^i \partial y^j}. \quad (2)$$

It will first be proved that the equation

$$\frac{dL(S_a)}{d\alpha} \Big|_{a=0} = \iint_R L'(S; x, y) \omega_a(x, y, 0) dx dy^* \quad (3)$$

is unaffected by an exceptional point of the first kind. A function $\Theta(x, y)$ of class $C^{(r)}$ will be chosen which vanishes excepting in the square

$$x_0 - \epsilon < x < x_0 + \epsilon, \quad y_0 - \epsilon < y < y_0 + \epsilon,$$

and is equal to unity in the square

$$x_0 - h \leq x \leq x_0 + h, \quad y_0 - h \leq y \leq y_0 + h,$$

where h is a positive number less than ϵ . The surface $S_{\epsilon a}$ will be defined by the equation

$$S_{\epsilon a}: \quad z = z(x, y) + \Theta(x, y) \omega(x, y, \alpha).$$

The surfaces S_a and $S_{\epsilon a}$ can then be made to lie in any neighborhood of S of order r by taking α sufficiently small. Since S_a coincides with $S_{\epsilon a}$ in the neighborhood of the exceptional point, the equation

$$\lim_{\alpha=0} \frac{L(S_a) - L(S_{\epsilon a})}{\alpha} = \iint_R L'(S; x, y) (1 - \Theta(x, y)) \omega_a(x, y, 0) dx dy,$$

is satisfied for all values of ϵ . It follows from equation (1) that

$$\lim_{\epsilon, \alpha=0} \frac{L(S_{\epsilon a}) - L(S)}{\alpha} = 0.$$

Since the region where $\Theta(x, y) \neq 0$ approaches zero with ϵ , these equations imply equation (3).

If equation (2) is satisfied at the point (x_0, y_0) , a new function will be defined by the equation

$$\bar{L}(S) = L(S) - \sum_{i,j=0}^n a_{ij} \frac{\partial^{i+j} z(x_0, y_0)}{\partial x^i \partial y^j}.$$

This function will then satisfy equation (1), and consequently equation (3) must be replaced by the equation

$$\frac{dL(S_a)}{d\alpha} \Big|_{\alpha=0} = \iint_R L'(S; x, y) \omega_a(x, y, 0) dx dy + \sum_{i,j=0}^n a_{ij} \frac{\partial^{i+j} \omega_a(x_0, y_0, 0)}{\partial x^i \partial y^j}. \quad (4)$$

It is easy to prove that if the surface S furnishes a minimum for the function $L(S)$, the constants a_{ij} must all be zero. It has already been proved that $L'(S; x, y) = 0$ is a necessary condition for a minimum.* Then, if $a_{kl} \neq 0$, the function $\omega(x, y, \alpha)$ can be defined as

$$\omega(x, y, \alpha) = \alpha (x - x_0)^k (y - y_0)^l$$

in the neighborhood of (x_0, y_0) , and equation (4) becomes

$$\frac{dL(S_a)}{d\alpha} = a_{kl} k! l! \neq 0,$$

and there can be no minimum.

An exceptional point will often occur if a function of a surface is differentiated twice at the same point. For instance, the function $L(S)$ can be defined as $L(S) = \phi(J)$, where ϕ is a function of class C'' , and

$$J = \iint_R f(x, y, z, p_{10}, p_{01}, \dots, p_{mm}) dx dy.$$

The argument p_{ij} is used to represent $\partial^{i+j} z / \partial x^i \partial y^j$. It can then be easily proved that the derivative

$$L'(S; x_0, y_0) = \phi'(J) E(x_0, y_0, z(x_0, y_0), \dots)$$

is continuous and approached uniformly with order $4m$, where

$$E(x, y, z, \dots) = \sum_{i,j=0}^m (-1)^{i+j} \frac{\partial^{i+j} f_{p_{ij}}}{\partial x^i \partial y^j}. \dagger$$

If the second derivative is taken at a different point it is seen to be

$$L''(S; x_0, y_0; x, y) = \phi''(J) E(x_0, y_0, \dots) E(x, y, \dots),$$

but (x_0, y_0) is now an exceptional point. At (x_0, y_0) it is found that

$$\lim_{\epsilon=0} \frac{L'(S_\epsilon; x_0, y_0) - L'(S; x_0, y_0)}{\alpha} = \phi'(J) \sum_{i,j=0}^{2m} \frac{\partial E}{\partial p_{ij}} \lim_{\epsilon=0} \frac{1}{\alpha} \cdot \frac{\partial^{i+j} \eta(x_0, y_0)}{\partial x^i \partial y^j}.$$

This is in the same form as equation (2), and consequently for this case

$$a_{ij} = \phi'(J) \frac{\partial E}{\partial p_{ij}}.$$

If $L(S)$ is a function whose derivatives $L'(S; x, y)$ and $L''(S; x_0, y_0; x, y)$ are continuous and approached uniformly excepting when $x = x_0$ and $y = y_0$, and if, in this case, equation (2) is satisfied, the second derivative of $L(S_a)$ with respect to α is found to be

* Fischer, *loc. cit.*, p. 295.

† Compare with Fischer, *loc. cit.*, p. 297.

$$\begin{aligned} \frac{d^2 L(S_a)}{d\alpha^2} \Big|_{\alpha=0} = & \iint_R \left\{ \iint_R L''(S; x_0, y_0; x, y) \omega_a(x_0, y_0, 0) \omega_a(x, y, 0) dx dy \right. \\ & + \sum_{i,j=0}^n a_{ij}(x_0, y_0) \frac{\partial^{i+j} \omega_a(x_0, y_0, 0)}{\partial x^i \partial y^j} \omega_a(x_0, y_0, 0) \\ & \left. + L'(S; x_0, y_0) \omega_{aa}(x_0, y_0, 0) \right\} dx_0 dy_0. \end{aligned} \quad (5)$$

The Jacobi condition for a minimum of a double integral can be derived from a special case of equation (5). If $L(S)$ is the double integral J , the second derivative $L''(S; x_0, y_0; x, y)$ vanishes. Then, if $\omega(x, y, \alpha) = \alpha \zeta(x, y)$, equation (5) becomes

$$\frac{d^2 L(S_a)}{d\alpha^2} = \iint_R \sum_{i,j=1}^{2m} \frac{\partial E}{\partial p_{ij}} \frac{\partial^{i+j} \zeta(x, y)}{\partial x^i \partial y^j} \cdot \zeta(x, y) dx dy.$$

The equation

$$\sum_{i,j=0}^{2m} \frac{\partial E}{\partial p_{ij}} \cdot \frac{\partial^{i+j} \zeta}{\partial x^i \partial y^j} = 0$$

is the analogue of the Jacobi equation for this problem.* If it has a solution, not identically zero, which vanishes along the boundary of R , the second variation of J can be made to vanish. The question whether it can also be made negative or not has not been settled yet, excepting in special cases.

§ 2. *Exceptional Curves.*

Two kinds of exceptional curves will also be considered. The first kind will be those along which

$$\lim_{\epsilon \rightarrow 0} \frac{L(S_\epsilon) - L(S)}{\epsilon \alpha} = 0 \quad (6)$$

uniformly in the neighborhood of the given surface. The second kind will be those for which there is a set of continuous functions $a_j(s)$ such that

$$\lim_{\epsilon \rightarrow 0} \frac{L(S_\epsilon) - L(S)}{\epsilon \alpha} = \sum_{j=0}^n \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \alpha} \int_{s-\epsilon}^{s+\epsilon} a_j(s) \cdot \frac{\partial^j \eta(s, 0)}{\partial n^j} ds, \quad (7)$$

where the variation $\eta(s, n)$ is expressed in terms of the length of arc along the exceptional curve and the normal distance to it. This limit is also assumed to be approached uniformly.

If the derivative $L'(S; x, y)$ is continuous and approached uniformly, excepting along a curve

$$C: \quad x=x(s), \quad y=y(s), \quad (a \leq s \leq b),$$

of class $C^{(r)}$, and equation (6) is satisfied along C , and if the function $\omega(x, y, \alpha)$ vanishes at the end-points of C , it will be proved that equation (3) is valid. A function $\Theta(x, y)$ of class $C^{(r)}$ can be chosen, which is equal to zero at every

*Bolza, "Vorlesungen über Variationsrechnung," p. 675.

point whose distance from C is greater than or equal to ε , and equal to unity at every point whose distance from C is less than or equal to h , a positive constant less than ε . The interval $a \leq s \leq b$ will then be divided into m equal parts by the points $s_0=a, s_1, \dots, s_m=b$, choosing m in such a way that $\frac{\varepsilon}{2} < s_1 - s_0 < \varepsilon$. The functions $\beta_i(x, y)$, also of class $C^{(r)}$, will be chosen in such a way that the equations

$$\begin{aligned}\beta_0(x(s), y(s)) &= 0, & (a \leq s \leq b), \\ \beta_i(x(s), y(s)) &= 1, & (a \leq s \leq s_i; i=1, 2, \dots, m), \\ 1 > \beta_i(x(s), y(s)) &> 0, & (s_i < s < s_{i+1}), \\ \beta_i(x(s), y(s)) &= 0, & (s_{i+1} \leq s \leq b),\end{aligned}$$

are satisfied. The surfaces $S^{(i)}$ will then be defined by the equations

$$S^{(i)}: \quad z = z(x, y) + \beta_i(x, y)\Theta(x, y)\omega(x, y, \alpha).$$

Since the surface S_a coincides with $S^{(m)}$ in the neighborhood of C and the hypothesis of the Volterra theorem is satisfied in the rest of the region R , the equation

$$\lim_{\varepsilon=0} \frac{L(S_a) - L(S^{(m)})}{\alpha} = \iint_R L'(S; x, y) (1 - \Theta(x, y)) \omega_a(x, y, 0) dx dy \quad (8)$$

must be satisfied. The surfaces $S^{(i+1)}$ and $S^{(i)}$ can now take the place of S_ε and S in equation (6). Since this limit is approached uniformly, there is a quantity ξ , which approaches zero with ε , such that

$$\left| \frac{L(S^{(i+1)}) - L(S^{(i)})}{\alpha} \right| < \xi \varepsilon, \quad (i=0, 1, \dots, m).$$

Adding these inequalities,

$$\left| \frac{L(S^{(m)}) - L(S)}{\alpha} \right| < m \xi \varepsilon < 2 \xi (b-a).$$

Consequently,

$$\lim_{\varepsilon=0} \frac{L(S^{(m)}) - L(S)}{\alpha} = 0. \quad (9)$$

Since the region where $\Theta(x, y) \neq 0$ approaches zero with ε , equations (8) and (9) imply equation (3), which was to be proved.

If equation (7) is satisfied along the curve C , the new function

$$\bar{L}(S) = L(S) - \sum_{j=0}^n \int_a^b a_j(s) \frac{\partial^j z(x(s), y(s))}{\partial n^j} ds$$

will satisfy equation (6), and consequently equation (3) must be replaced by

$$\frac{dL(S_a)}{d\alpha} \Big|_{\alpha=0} = \iint_R L'(S; x, y) \omega_a(x, y, 0) + \sum_{j=0}^n \int_a^b a_j(s) \frac{\partial^j \omega_a}{\partial n^j} ds.$$

A necessary condition for a minimum can be derived from this equation. It follows immediately that, if there is a minimum, the equation

$$\sum_{j=0}^n \int_a^b a_j(s) \frac{\partial^j \omega_a}{\partial n^j} ds = 0 \quad (10)$$

must be satisfied for every choice of $\omega(s, n, \alpha)$. This will be proved to be equivalent to the equations

$$a_j(s) = 0, \quad (j=0, 1, \dots, n; a \leq s \leq b). \quad (11)$$

Suppose that $a_l(s_1) \neq 0$. Then, since the functions $a_j(s)$ are assumed to be continuous, there is an interval including s_1 in which $a_l(s) \neq 0$. Let $\Theta(s)$ be an arbitrary function of class $C^{(r)}$ which agrees in sign with $\omega_l(s)$ in such an interval and vanishes outside of it. Then the function $\omega(s, n, \alpha)$ will be defined in the neighborhood of C by the equation

$$\omega(s, n, \alpha) = \alpha n^l \Theta(s).$$

Consequently,

$$\frac{\partial^j \omega_a(s, n, 0)}{\partial n^j} = 0, \quad (j \neq l),$$

and

$$\int_a^b a_l(s) \frac{\partial^l \omega(s, n, 0)}{\partial n^l} ds > 0,$$

and equation (10) is not satisfied. Therefore, equations (11) form a necessary condition that the surface S minimize the function $L(S)$.

§ 3. *The Variable Boundary Problem.*

In the work up to this point it has been assumed that the surfaces S_a were all bounded by the same space curve. If this is not the case, as, for instance, if the locus of the bounding curves is a given surface, the surfaces S_a will be supposed to extend beyond this boundary and all intersect along a new space curve. The function $L(S)$ will depend for its value only on the part of S inside the original boundary. Thus, the function $z(x, y)$ will be defined arbitrarily in a region R' surrounding R , and the function $\omega(x, y, \alpha)$ defined so as to vanish along the boundary of the region $R + R'$. Under these circumstances the derivative $L'(S; x, y)$ will vanish in R' and very likely be infinite along the curve separating R' from R . If the function considered is the integral J considered in § 1, the boundary of R is an exceptional curve of the last kind mentioned, and the functions $a_j(s)$ will now be proved to exist. It will be supposed that the surfaces over which the double integral is taken are bounded by a surface

$$\tilde{S}: \quad z = \tilde{z}(x, y).$$

The curve C will be the projection of the intersection of $\tilde{S}$ and S , and the func-

tion $\eta(x, y)$ chosen so as to define the derivative $L'(S; x_0, y_0)$, where (x_0, y_0) is an arbitrary point of C . The distance n will be taken positive inside the region R , and the positive direction of s chosen so that $x_s y_n - x_n y_s = 1$. The value of n on the projection of C_ϵ of the intersection of the surfaces S_ϵ and $\tilde{S}$ will be called $N(s)$.

Then the equations

$$\left. \begin{aligned} \tilde{z}(x(s, N(s)), y(s, N(s))) - z(x(s, N(s)), y(s, N(s))) &= \eta(x(s, N(s)), y(s, N(s))), \\ \tilde{z}(x(s, 0), y(s, 0)) - z(x(s, 0), y(s, 0)) &= 0, \end{aligned} \right\} \quad (12)$$

must be satisfied. Subtracting and applying Taylor's formula,

$$((\tilde{p} - p)x_n + (\tilde{q} - q)y_n)N(s) = \eta(x(s, N(s)), y(s, N(s))),$$

where the arguments of $\tilde{p}, p, \tilde{q}$ and q are $x(s, \theta N(s)), y(s, \theta N(s)), (0 < \theta < 1)$. Consequently,

$$\lim_{\epsilon=0} \frac{N(s)}{\eta} = \frac{1}{(\tilde{p} - p)x_n + (\tilde{q} - q)y_n}. \quad (13)$$

If the last of equations (12) is differentiated with respect to s , it becomes

$$(\tilde{p} - p)x_s + (\tilde{q} - q)y_s = 0.$$

Consequently, equation (13) may be written

$$\lim_{\epsilon=0} \frac{N(s)}{\eta} = \frac{-y_s}{\tilde{p} - p} = \frac{x_s}{\tilde{q} - q}.$$

This expression is finite unless $\tilde{S}$ is tangent to S . It follows from the definitions of S_ϵ and $L(S)$ that

$$L(S_\epsilon) - L(S) = \iint_{\Delta S} \sum_{i,j=0}^n f_{ij}(x, y, z + \theta\eta, \dots) \eta_{ij} dx dy - \int_{x(s,0)}^{x(s,N(s))} \int_{y(s,0)}^{y(s,N(s))} f(x, y, z + \eta, \dots) dx dy, \quad (14)$$

where

$$f_{ij} = f_{p_{ij}}, \quad \eta_{ij} = \frac{\partial^{i+j}\eta}{\partial x^i \partial y^j} = \frac{\partial^{i+j}\eta(s, n)}{\partial s^{i+j}} \left(\frac{\partial s}{\partial x}\right)^i \left(\frac{\partial s}{\partial y}\right)^j + \dots + \frac{\partial \eta}{\partial n} \frac{\partial^{i+j}n}{\partial x^i \partial y^j}, \quad (15)$$

and ΔS is the part of R where $\eta \neq 0$.

The first integral of equation (14) can be reduced to the expression

$$\iint_{\Delta S} E(x, y, z + \theta\eta, \dots) \eta dx dy + \int_{s_0-\epsilon}^{s_0+\epsilon} \left\{ y_s \sum_{i=1}^n \sum_{j=0}^i (-1)^{k-1} \frac{\partial^{k-1} f_{ij}}{\partial x^{k-1}} \eta_{i-kj} - x_s \sum_{i=0}^n \sum_{j=1}^j (-1)^{i+l-1} \frac{\partial^{i+l-1} f_{ij}}{\partial x^i \partial y^{l-1}} \eta_{0j-l} \right\} ds,$$

by repeated application of Green's theorem.*

The last integral in equation (14) is equal to

$$\int_{s_0-\epsilon}^{s_0+\epsilon} f(x, y, z + \eta, \dots) N(s) ds,$$

* Bolza, *loc. cit.*, p. 654.

where $x = x(s, \theta N(s)), \dots$. If these expressions are substituted in equation (14), and it is divided by $\epsilon\alpha$, and then the limit taken as $\epsilon \rightarrow 0$ the double integral vanishes because ΔS is of the second degree in ϵ and η/α is finite, and the equation becomes

$$\lim_{\epsilon \rightarrow 0} \frac{L(S_\epsilon) - L(S)}{\epsilon\alpha} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon\alpha} \sum_{i,j=0}^n \int_{s_0-\epsilon}^{s_0+\epsilon} b_{ij}(s) \eta_{ij} ds, \quad (16)$$

where

$$\begin{aligned} b_{00} &= \frac{y_s}{\tilde{p}-p} \left[f + (\tilde{p}-p) \sum_{k=1}^n (-1)^{k-1} \frac{\partial^{k-1}}{\partial x^{k-1}} f_{k0} + (\tilde{q}-q) \sum_{i=0, l=1}^n (-1)^{i+l-1} \frac{\partial^{i+l-1} f_{il}}{\partial x^i \partial y^{l-1}} \right], \\ b_{0j} &= \frac{y_s}{\tilde{p}-p} \left[(\tilde{p}-p) \sum_{k=1}^n (-1)^{k-1} \frac{\partial^{k-1}}{\partial x^{k-1}} f_{kj} + (\tilde{q}-q) \sum_{i=0}^n \sum_{l=1}^{n-j} (-1)^{i+l-1} \frac{\partial^{i+l-1} f_{ij+l}}{\partial x^i \partial y^{l-1}} \right], \\ &\quad (j=1, 2, \dots, n-1), \\ b_{ij} &= y_s \sum_{k=1}^{n-i} (-1)^{k-1} \frac{\partial^{k-1} f_{i+kj}}{\partial x^{k-1}}, \quad (i=1, 2, \dots, n-1; j=0, 1, \dots, n). \end{aligned}$$

If equations (15) are substituted in equation (16), it takes the form

$$\lim_{\epsilon \rightarrow 0} \frac{L(S_\epsilon) - L(S)}{\epsilon\alpha} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon\alpha} \sum_{i,j=0}^n \int_{s_0-\epsilon}^{s_0+\epsilon} b'_{ij}(s) \frac{\partial^{i+j} \eta(s, 0)}{\partial s^i \partial n^j}. \quad (17)$$

Since the partial derivatives of η vanish at $s_0 \pm \epsilon$, the equations

$$\int_{s_0-\epsilon}^{s_0+\epsilon} b'_{ij}(s) \frac{\partial^{i+j} \eta(s, 0)}{\partial s^i \partial n^j} ds = - \int_{s_0-\epsilon}^{s_0+\epsilon} \frac{db'_{ij}(s)}{ds} \cdot \frac{\partial^{i+j-1} \eta(s, 0)}{\partial s^{i-1} \partial n^j} ds$$

must be satisfied. If this process is repeated often enough equation (17) is reduced to the form of equation (7).

If f is a function of x, y, z, p, q only, the only one of the functions $a_j(s)$ which occurs is $a_0(s)$. It is found to be

$$a_0(s) = \frac{y_s}{\tilde{p}-p} (f + (\tilde{p}-p)f_p + (\tilde{q}-q)f_q).$$

The vanishing of this expression constitutes the variable boundary condition for a minimum of such a double integral. If f depends on the partial derivatives of the second order also, the functions $a_0(s)$ and $a_1(s)$ occur. They are found to be

$$\begin{aligned} a_0(s) &= \frac{y_s}{\tilde{p}-p} f + f_{10} y_s - f_{01} x_s - \frac{\partial f_{20}}{\partial x} y_s (1 + x_s y_n) - \frac{\partial f_{20}}{\partial y} y_s^2 y_n + \frac{\partial f_{11}}{\partial x} x_s^2 y_n \\ &\quad + \frac{\partial f_{11}}{\partial y} y_s^2 x_n - \frac{\partial f_{02}}{\partial x} x_s^2 x_n + \frac{\partial f_{02}}{\partial y} x_s (1 - y_s x_n) - f_{20} (y_{ss} y_n + y_s y_{sn}) \\ &\quad + f_{11} (y_{ss} x_n + y_s x_{sn}) - f_{02} (x_{ss} x_n + x_s x_{sn}), \\ a_1(s) &= -f_{20} y_s^2 + f_{11} x_s y_s - f_{02} x_s^2. \end{aligned}$$

If the given double integral depends on derivatives of higher order, the functions $a_j(s)$ can be computed in the same way, but the work is long and will not be given here.